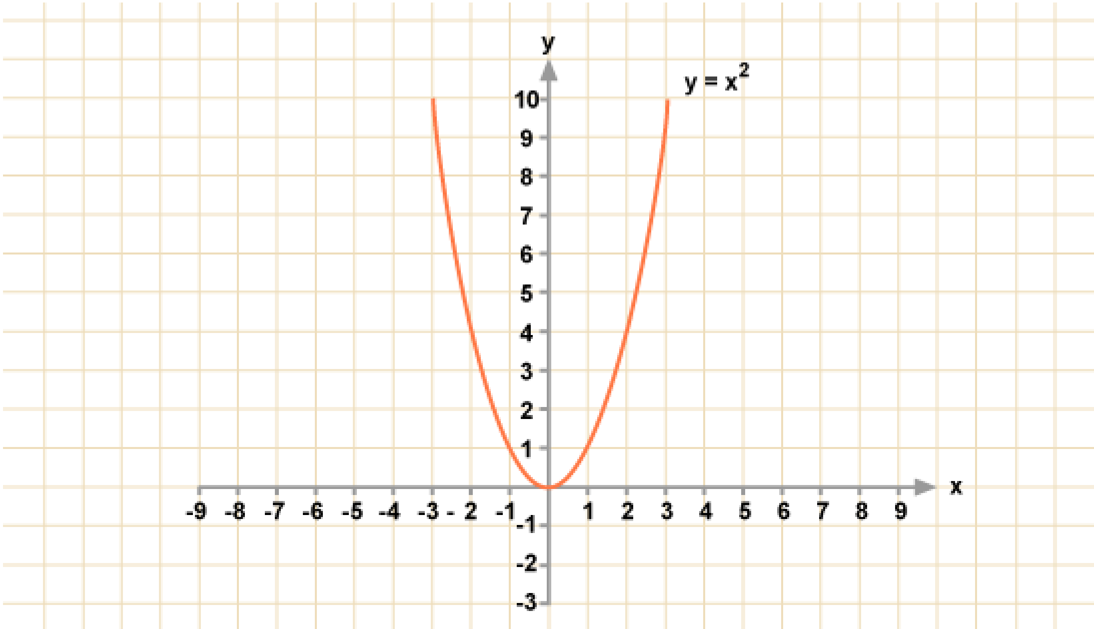
Mini-Investigation (1): Differentiation and its Applications (Calculus Part One)

This essay is my Mini-Investigation topic for the Michaelmas and first two weeks of the Lent term. This is not really an 'investigation' as such. During my homework slot, I have learnt new things and this is basically a summary of what I have learnt. It is split into 'Sections' which loosely follow each week's work (This is somewhat inaccurate due to the fact that for some weeks, I did more than others). My Topic of Research is about Differentiation and its applications. Differentiation is one of the two key ideas of calculus (A very interesting branch of Mathematics). The other key idea is Integration (I will be studying this for my second Mini-Investigation this year). This topic is a follow up of Algebra and Functions of sorts. I will start by explaining key Ideas before moving on to harder applications.

Section 1- Limits

To start, I will explain what Limits are, as these are useful in Differentiation. Consider the graph: $y = x^2$



We use Limits to detonate the limit value of the function as x approaches a certain point. Here is an example:

$$\lim_{x \to 2} x^2 = 4$$

This simple means that as x approaches two, the limit of the function is 4. A way we can understand this is by understanding the difference between f(2) and the limit of x approaching 2. f(2) Is the value as it equals two, but the limit is as its approaching two (the maximum value it can be as it is approaching 2). This seems useless to us now, but it is actually very useful.

This same concept can be applied to sequences as well. Actually, we can use it to generalise problems which we are unable to sensibly solve. For example:

What is the value of $\frac{1}{\infty}$?

This is very hard to determine. If we turn it into a word problem, it will be something like:

"If I have one apple, and I want to split it into infinitely many portions, how many portions of an apple will there be in each group?"

Well, there is no way to find out, but we can **give a limit value!**

Consider this table:

x	$1/x$
1	1
2	0.5
3	0.33333

Can you see that when x gets larger and larger (i.e. approaches infinity), $1/x$ actually gets closer and closer (tends to) 0? So this therefore means:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

- There is no way to actually solve this problem
- However, we can generalise using limits.

Lets study one more limit problem involving infinity (Solve the following):

$$\lim_{x\to\infty} 2x$$

This problem is easier because now, if we apply the same thing as last time, as x tends to infinity, $2x$ must also approach infinity as it is just x multiplied by two!

So the answer is:

$$\lim_{x\to\infty} 2x = \infty$$

So now, from something we thought had no purpose, we can learn that it does serve a very big purpose. It also is used a lot in calculus as you will slowly begin to see.

Section 2- Average Rate of Change and Differential Coefficients

Let's study the following question to help us understand this:

"Find the slope of line AB, where points A and B are on the graph of $f(x) = x^2$ *and where the* x *coordinate of A is 1, and B is two units greater"*

- The first thing we should do is find out that the x coordinate of B is 3 (1+2 units greater = 3)

I will show the following formula and then we will study it. The formula to find the average rate of change as x varies from one number to another, can be expressed as this:

$$\frac{f(a+h) - f(a)}{h}$$

- a is the initial point
- h is by how much it has increased
- Instead of dividing by h, we can also divide by (b-a). But remember, both will tell you **how much it increases** (b-a is the difference, so also the distance between or how much it has increased)

So I should substitute the values that I have into this:

$$\frac{f(1+2) - f(1)}{3-1}$$

Ok, so now I deduce that it is:

$$\frac{f(3) - f(1)}{2}$$

Because the function is $f(x) = x^2$:

$$\frac{3^2 - 1^2}{2}$$

And this equals 4. Therefore, the average rate of change as x varies from 1 to 3 is actually 4.

Let's say that A is 1 and B is h units greater, let's plug it into the formula:

$$\frac{f(h+1) - f(1)}{h}$$

$$\frac{h^2 + 2h + 1 - 1}{h}$$

$$\frac{h^2 + 2h}{h}$$

Dividing by h,

$$h + 2$$

Here, cancel out h as h$\neq$ 0 (Because we divided by h)

Answer: 2

However! Just because h is not 0, it can still be other values! Consider this table:

h	The Slope
1	3
0.5	2.5
0.25	2.25

Do you notice that as h tends to 0, the slope of AB tends to 2?

h cannot be 0, but it can approach zero! This is where we can use limits:

$$\lim_{h \to 0} h + 2 = 2$$

Given that h cannot be 0, we now have the final rate of change formula:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So for function f(a), f'(a) (f prime of a) is the above formula. f'(a) is called the differential coefficient at x=a (Or the rate of change).

So let's look at an example.

"for the function f(x)=x^2, calculate the differential coefficient at x=3"

$$f'(3) = \quad \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$\lim_{h \to 0} \frac{h^2 + 6h}{h}$$

So can you see that the limit is 6?

Let us try it with $f'(x)$:

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x\hat{}2}{h}$$

$$\lim_{h \to 0} 2x + h = 2x$$

$2x$ is known as the derivative of x^2

Sometimes, in a maths exam, you might see another notation being used to find a derivative. In fact, the rate of change formula actually means:

$$\frac{incremental\ change\ in\ y}{incremental\ change\ in\ x}$$

$f(x)$ is actually another way of writing the y coordinate. This means that $f(a+h) - f(a)$ is change in y. h is also the change in x because of the fact I mentioned earlier about (b-a) being used to replace h.

People can write it like this:

$$\frac{\Delta y}{\Delta x}$$

The triangle means delta, which means "change in." People can also write in like this:

$$\frac{dy}{dx}$$

You might be asked to find $\frac{dy}{dx}$ of a curve. This just means the derivative.

<u>Section 3- Differentiating Different Kinds of Functions:</u>

Instead of using the rate of change formula to find derivatives, Mathematicians have found an easy way out. This is the rule:

"If n is a natural number, then:"

$$\frac{dy}{dx}(x^n) = nx^{n-1}$$

For example:

$$\frac{dy}{dx}(x^2) = 2x$$

$$\frac{dy}{dx}(3x^4) = 12x^3$$

$$\frac{dy}{dx}(3x) = 3$$

If we differentiate an integer, we will get 0 (This is because you should imagine that it is nx^0)

However, there are many different types of functions; we want to know the derivative of all of them. I will first put up my findings, then my explanations (Due to a word limit, I will only cover exponential and logarithmic functions):

$\boldsymbol{f(x)}$	$\boldsymbol{f'(x)}$
e^x	e^x
a^x	$a^x \ln(a)$
$\ln(x)$	$1/x$

Firstly, we want to prove what the derivative of ln(x) is. We are going to begin by using the rate of change formula to prove it:

$$\lim_{\mathrm{d}x \to 0} \left(\frac{\ln(x+dx) - \ln(x)}{dx}\right)$$

Given that $\log(a) - \log(b) = \log(\frac{a}{b})$,

$$\lim_{dx \to 0} \frac{\ln(\frac{x+dx}{x})}{dx}$$

$$\lim_{dx \to 0} \frac{1}{dx} \times \ln(1 + \frac{dx}{x})$$

Given that $nlog(a) = \log(a^n)$

$$\lim_{dx \to 0} \ln(1 + \frac{dx}{x})\text{^} 1/dx$$

Let us assume that $k = \frac{dx}{x}$ and that $\frac{1}{kx} = \frac{1}{dx}$ $(kx = dx)$ as well:

$$\lim_{k \to 0} \ln((1+k)\text{^}\frac{1}{k} \times \frac{1}{x})$$

Rearranging:

$$\frac{1}{x} \ln(\lim_{k \to 0} (1+k)^{\frac{1}{k}})$$

Do you notice how the bracketed expression equals e? Think back to compound interest! Substitute k for 1/n ! If we solve the limit, we will get this:

$$\frac{1}{x} \ln(e)$$

Given that ln(e)= 1:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

So now, using this, we will find the derivative of e^x:

$$\frac{d}{dx}(\ln(e^x)$$

$$\frac{d}{dx}(x \times \ln(e)$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}(\ln e\hat{}x) = \frac{1}{e^x}$$

Above, I have substituted $x$with e^x

$$\frac{1}{e^x} = 1$$

$$1 = e^x$$

We have used two rules to find the derivative of one function. We firstly found that the derivative of x is 1. We then substituted our function with x, which told us what the derivative is:

$$\frac{d}{dx}e^x = e^x$$

Finally, using this we can find the derivative of a^x:

$$\frac{d}{dx}(a^x)$$

We should rewrite this in a way we already know:

$$e^{xln(a)}$$

I am now going to apply a chain rule. We are going to obtain a derivative of function with respect to the exponent and then multiply it by the derivative of the exponent with respect to x. This is what we get:

$$e^{xln(a)} \times \ln(a)$$

Now we will make x the second exponent,

$$e^{\ln(a)^x}$$

If you raise e to the natural log of something, it just is that something, so this now means that:

$$\frac{d}{dx}(a^x) = a^x \times \ln(a)$$

Section 4- Tangents and Slopes:

The derivative of a function is the gradient or the slope of the tangent at any given point (x,y).

If I want to find the equation of that tangent, this is the formula:

$$y - f(a) = f'(a)(x - a)$$

Let us first define a tangent. A tangent is a line that comes and intersects with a curve at only one point. Think of it like going off on a tangent when you speak, you begin on topic, but then you end up talking about something completely irrelevant. The derivative of a function is the gradient (steepness) of the tangent (If you understand straight line graphs, this is quite simple). Let us say that I want to calculate the tangent of the curve y=x^3 at point (1,1), these are the steps I would do:

$$\frac{dy}{dx} = 3x^2$$

$$f(1) = 3$$

You substitute x for the x coordinate of the point of tangency. We now know that the slope is 3.

Using the formula,

$$y - 1 = 3(x - 1)$$

Therefore, the equation of the line is $y = 3x - 2$

The End

Thank you very much for reading my mini-investigation. I hope that by reading this, your mathematical interest and knowledge has increased. In my second mini-investigation, I will study Integral Calculus to completely understand the basics of calculus. I believe that maths is an incredibly under-rated subject and should be appreciated by everyone, everywhere. By proving formulas, you also continue to prove the beauty of mathematics and prove how wonderful it is.

Thank You.